Combating Alien Plants Rooted in the Heel of a Dam: Injaka Dam

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Contents

Introduction

- Dams and reservoirs are crucial infrastructure for various water usages.
- Challenge of dam systems being affected by IAS.
- These plants pose a risk to infrastructure, increase water loss and negatively affect water quality.
- The Injaka Dam, in Bushbuckridge (Mpumalanga), formed by the Mgwaritjie and Mgwaritjana Rivers, is affected by this problem

Problem statement

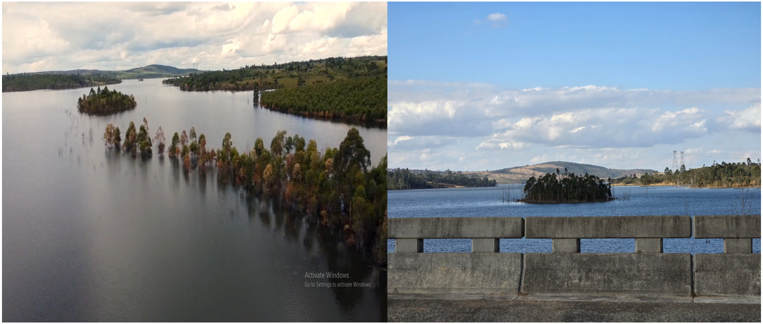


Figure: Injaka dam

Cont.

- Trees above the water surface obstruct navigation and pose immediate risks to boats and recreational activities.
- Submerged trees are often not visible, increasing the risk of vessel collisions.
- Submerged trees continue to grow and may eventually emerge, creating future hazards.

Motivation

- "Develop and implement a removal and remediation programme for alien invasive vegetation within a buffer of 30 m of the High Flood Line (shore line) of the dam in partnership with Working for Water."
- There is a notable lack of quantitative decision-support tools that guide how, when, and which plants should be removed under resource constraints. Most studies focus on descriptive assessments or post-intervention evaluations rather than predictive planning.

Objectives

- To develop a mathematical framework for early detection and optimal removal of invasive alien plants.
- To minimise ecological risks to aquatic life while supporting tourism and recreational use of water bodies.

Literature

- Studies have shown that up to 50% of annual flows could be used up by IAS if left uncleared.
- Findings in the literature have shown that IAS negatively impact the availability of water as they increase evapotranspiration.
- There are analytical models that optimizes prevention, eradication and control of IAS by balancing resources allocation across invasion stages.
- Variety of strategies have been employed to control IAS in aquatic environments. These include chemical(using herbicides), biological control(introducing natural enemies) and mechanical(physical removal) which is preferred in dam systems.

Mathematical Model

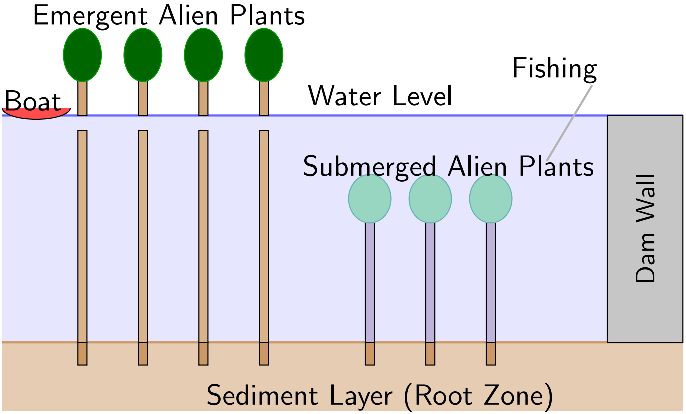


Figure: Schematic physical geometry of Injaka Dam.

Modeling Approach

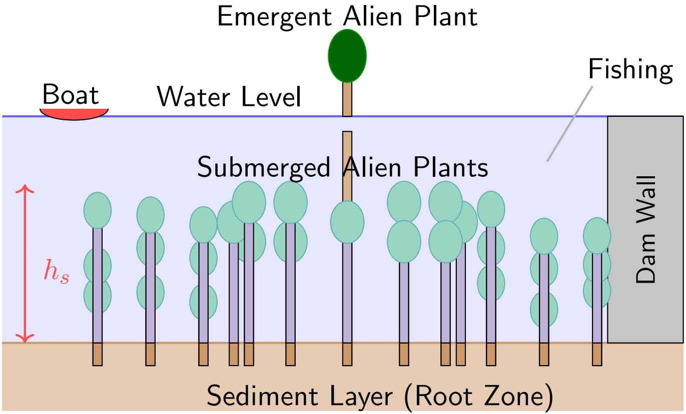
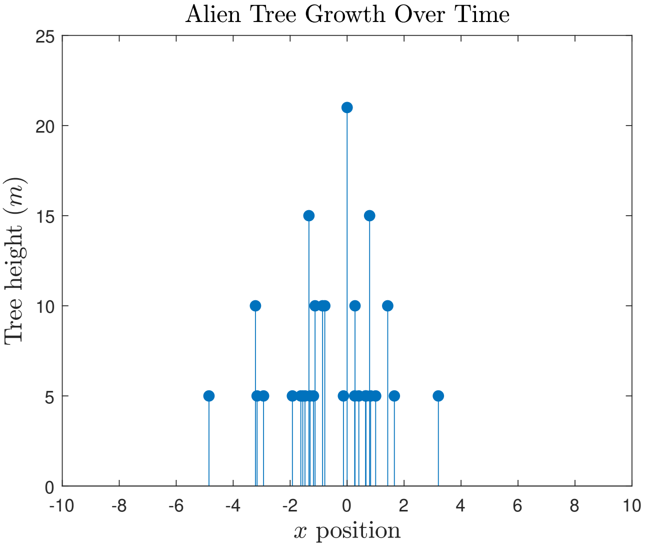


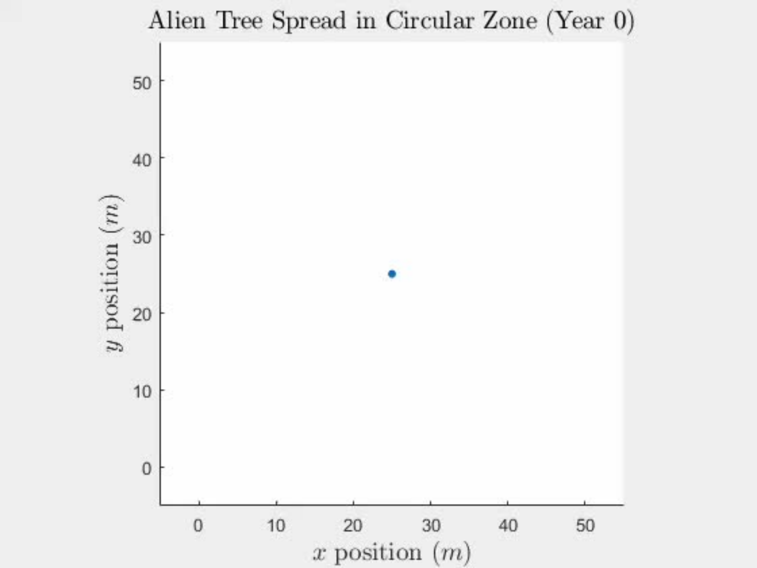
Figure: Schematic physical geometry of Injaka Dam with single emergent alien plant surrounded by submerged alien plants of height h_s.

Simulation of the Problem



Figure

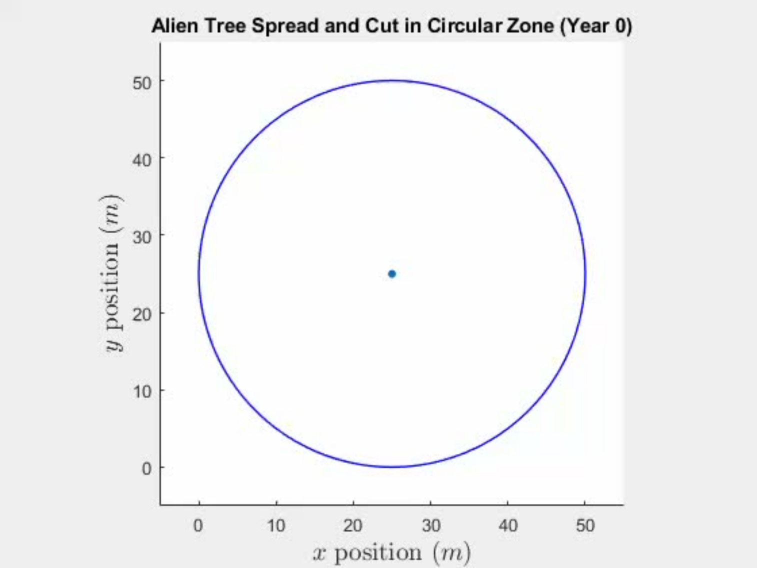
Cont.


Alien Tree Spread in Circular Zone (Year 0)
y position (m)
x position (m)
50
40
30
20
10
0
0
10
20
30
40
50

Summary

This simulation models the spatial spread of alien trees inside a circular dam. Each tree grows in height over time and begins producing seeds once its height exceeds a fixed threshold. Seeds are dispersed randomly within a prescribed radius, generating new trees whose positions remain inside the dam boundary. A management strategy is imposed by cutting trees that grow near the dam edge, thereby preventing further spread toward the boundary. The simulation evolves in yearly time steps and visualizes tree growth, reproduction, and cutting.

Cont.


Alien Tree Spread and Cut in Circular Zone (Year 0)
y position (m)
50
40
30
20
10
0
0
10
20
30
40
50
x position (m)

The optimal control model

The dynamics of the alien trees over a finite time interval $[0, T]$, with $T > 0$ is described by

$$\frac{dx}{dt} = f(x) - g(x)u(t), \quad t \in [0, T], \quad x(0) = x_0,$$

where:

- $x = x(t)$ - is the density of the population of alien trees;
- $u = u(t)$ - is the control effort of removal of alien trees;
- $f(x)$ - is the growth function of the population, model natural growth dynamics in the absence of control

  $$f(x) = rx(1 - \frac{x}{k}),$$

  r - is the growth rate,
  k - is the carrying capacity;
- $g(x)$ - the effectiveness of applying the control.

The optimal control model

The dynamics of how the average height of the alien trees changes is defined by

$$\frac{dy}{dt} = h(y) - G(x, y, u), \quad t \in [0, T], \quad y(0) = y_0,$$

where:

- y - is the average height of the alien trees;
- $h(y)$ - is the growth function of the tree height. We model the function using the *Chapman-Richards* model

  $$h(x) = my^p - ny,$$

  m - is the growth coefficient,
  p - is the allometric scaling,
  n - is the rate of senescence;

- $G(x, y, u)$ is the instantaneous shift in the average height caused by the removal of alien trees

  $$G(x, y, u) = \left[\frac{g(x)u}{x}\right](y_c - y),$$

  where y_c is average height of the trees being removed.

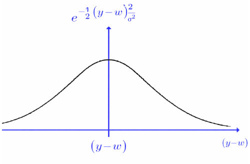
The optimal control model

The goal is to minimize the risk of accidents and the cost associated with the allocation of effort for species removal.

- Risk Model

$$R(x) = \beta\left(1 - e^{-\gamma x}\right) e^{-\frac{1}{2}\frac{(y-w)^2}{\sigma^2}}$$

where $y = y(t)$ characteristic the height of the trees and $w = w(t)$ is the water level.



- Direct cost $C(u)$ - cost associated with control effort. We assume the general quadratic form

$$C(u) = c_1 u - \frac{1}{2} c_2 u^2, \quad c_1, c_2 > 0.$$

- The final cost $\phi(x(T))$ - the terminal cost associated with the state of the population of the alien at the end

$$\phi(x(T)) = \frac{1}{2} x(T)^2.$$

The optimal control model

The optimal control model to efficiently remove alien trees in a dam is defined by

$$\min_u J(u) = \frac{1}{2}x(T)^2 + \int_0^T \left[\frac{1}{2}cu^2 + \beta(1-e^{-\gamma x})e^{-\frac{(y-w)^2}{2\sigma^2}}\right] dt,$$

subject to:

$$\frac{dx}{dt} = rx(1-\frac{x}{k}) - g(x)u(t), \quad x(0) = x_0,$$

$$\frac{dy}{dt} = my^p - ny - \left[\frac{g(x)u(t)}{x}\right](y_c - y), \quad y(0) = y_0.$$

$$u(t) \in L^2(0,T), \quad x(t) \geq 0, \quad y(t) \geq 0.$$

- $J(u)$ coercive in u (with $u(t) \geq 0$), ensuring an optimal control $u^*(t)$ exists under standard well-posedness assumptions.
- The quadratic control cost is strictly convex in u, so the Hamiltonian has a unique minimiser in u at each time t.

The optimal control model

Let u^* be an optimal control with state (x^*, y^*). Then there exist adjoint variables (λ_1, λ_2) such that Pontryagin's Maximum Principle yields:

- The optimal control satisfies the following conditions:

  $$u^*(t) = \arg\min_u H(t, x^*, u, \lambda),$$

  where $H(t, x, u, \lambda)$ is the Hamiltonian function of the problem, given by:

  $$\begin{aligned} H = \frac{1}{2}cu^2 + \beta(1 - e^{-\gamma x})e^{-\frac{(y-w)^2}{2\sigma^2}} + \lambda_1 \left[rx\left(1 - \frac{x}{K}\right) - ug(x) \right] \\ + \lambda_2 \left[(my^p - ny) - \frac{ug(x)}{x}(y_c - y) \right]. \end{aligned}$$

  Hence, the optimal control $u^*(t)$ that minimize the Hamiltonian is given by

  $$u^*(t) = \frac{g(x)}{c}\left[\lambda_1(t) + \frac{\lambda_2(t)}{x}(y_c - y) \right].$$

The optimal control model

Let u^* be an optimal control with state (x^*, y^*). Then there exist adjoint variables (λ_1, λ_2) such that Pontryagin's Maximum Principle yields:

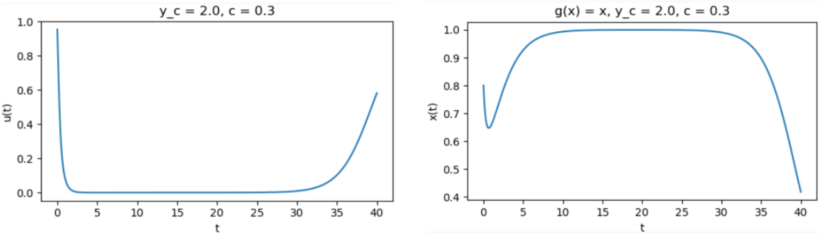
- The adjoint variables $\lambda_1(t)$ and $\lambda_2(t)$ satisfies the adjoint equation:

$$\frac{\lambda_1}{dt} = -\beta\gamma e^{-\gamma x} e^{-\frac{(y-w)^2}{2\sigma^2}} - \lambda_1 \left[r \left(1 - \frac{2x}{K} \right) - ug'(x) \right] + \lambda_2 \frac{u(y_{cut} - y)}{x} \left[g'(x) - \frac{g(x)}{x} \right]$$

$$\frac{\lambda_2}{dt} = \beta(1 - e^{-\gamma x}) \left(\frac{y-w}{\sigma^2} \right) e^{-\frac{(y-w)^2}{2\sigma^2}} - \lambda_2 \left[mpy^{p-1} - n + \frac{ug(x)}{x} \right]$$

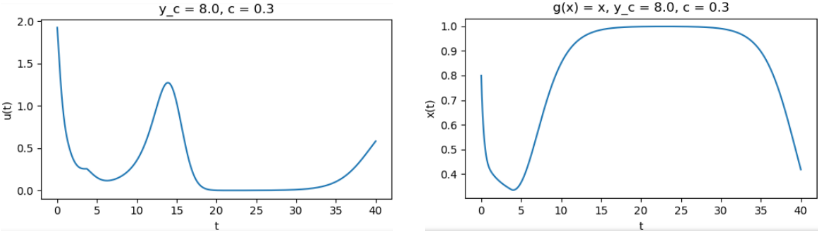
with the terminal condition $\lambda_1(T) = x^*(T)$ and $\lambda_2(T) = 0$.

Comparison of Control and State Variables



(a) Control Variable (Case 1)

(b) State Variable (Case 1)

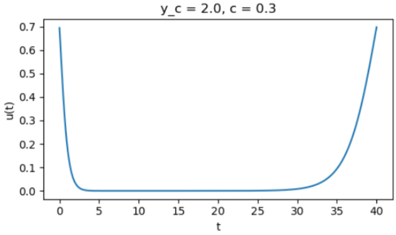


(c) Control Variable (Case 2)

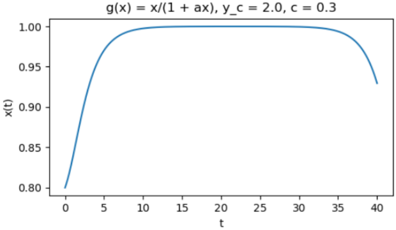
(d) State Variable (Case 2)

Figure: Comparison of Control and State Variables

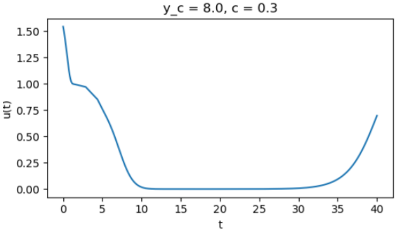
Comparison of Control and State Variables



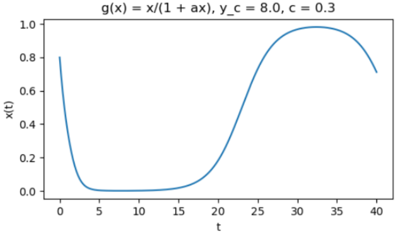
(a) Control Variable (Case 1)



(b) State Variable (Case 1)



(c) Control Variable (Case 2)



(d) State Variable (Case 2)

Conclusion and future work

- Results show that early, targeted removal reduces ecological risk and improves dam safety while minimising costs.
- The framework provides a quantitative decision-support tool for sustainable dam and ecosystem management.
- Zoning Method